Journal of Number Theory 219 (2021) 445–459

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Computational Section

Monomials of Eisenstein series

Trevor Griffin [a], Nathan Kenshur [b], Abigail Price [c], Bradshaw Vandenberg-Daves [d], Hui Xue [e], Daozhou Zhu [e,*]

[a] *University of Idaho, Moscow, ID, 83844, United States of America*
[b] *UC Berkeley, Berkeley, CA, 94720, United States of America*
[c] *Hillsdale College, Hillsdale, MI, 49242, United States of America*
[d] *Vassar College, Poughkeepsie, NY, 12604, United States of America*
[e] *School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC, 29634, United States of America*

A R T I C L E I N F O

Article history:
Received 6 July 2020
Accepted 30 September 2020
Available online 21 October 2020
Communicated by L. Smajlovic

MSC:
primary 11F11
secondary 11F03, 11F67

Keywords:
Eisenstein series
Zeros
Special values of L-functions

A B S T R A C T

Let $E_k(z)$ be the normalized Eisenstein series of weight k for the modular group $\mathrm{SL}(2,\mathbb{Z})$. We study the zeros of E_k to prove that the equation

$$\prod_{i=1}^{n} E_{k_i} = \prod_{j=1}^{m} E_{\ell_j}$$

has no solutions, except for those given by known relationships between E_4, E_6, E_8, E_{10}, and E_{14}. We go on to discuss some implications of this result.

* Corresponding author.
E-mail addresses: grif1718@vandals.uidaho.edu (T. Griffin), nkenshur@berkeley.edu (N. Kenshur), aprice1@hillsdale.edu (A. Price), bvandenbergdaves@vassar.edu (B. Vandenberg-Daves), huixue@clemson.edu (H. Xue), zdaozho@clemson.edu (D. Zhu).

https://doi.org/10.1016/j.jnt.2020.09.002
0022-314X/

1. Introduction

Let

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

be the normalized Eisenstein series of weight k over $\mathrm{SL}(2, \mathbb{Z})$, where $k \geq 4$ is an even integer, B_k is the kth Bernoulli number, and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Several previous papers have studied the relationships between products of Eisenstein series. Using the Rankin-Selberg method, Duke [Duk99] and Ghate [Gha00] each independently proved that the equation

$$E_{k_1} E_{k_2} = E_\ell$$

has only solutions forced by dimension considerations, i.e. those given by

$$E_4^2 = E_8, \quad E_4 E_6 = E_{10}, \quad E_4 E_{10} = E_{14}, \quad E_6 E_8 = E_{14}. \tag{1.1}$$

Emmons and Lanphier [EL07] extended this result to the case

$$\prod_{i=1}^{n} E_{k_i} = E_\ell,$$

proving that this equation has solutions only for $\ell \in \{8, 10, 14\}$, where these solutions are among the list given in equation (1.1). Their proof relies on controlling the growth of the coefficients

$$C_k = \frac{(2\pi i)^k}{\zeta(k)(k-1)!} = -\frac{2k}{B_k},$$

using their rapid decrease in magnitude to argue that the q-coefficients of $\prod_{i=1}^{n} E_{k_i}$ and E_ℓ cannot be equal in general.

In a somewhat different direction, Nozaki [Noz08] studied the function

$$F_k(\theta) = e^{ik\theta/2} E_k(e^{i\theta}) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c,d \in \mathbb{Z}}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k} = 2\cos(k\theta/2) + T_k(\theta), \tag{1.2}$$

first considered in [RS70], where $T_k(\theta)$ becomes trivially small as k increases. The functions $F_k(\theta)$ and $E_k(e^{i\theta})$ have the same zeros in $[\pi/2, 2\pi/3]$, allowing Nozaki to approximate locations of the zeros of Eisenstein series using the zeros of $2\cos(k\theta/2)$.

In this paper, we refine Nozaki's methods to demonstrate that for any $\ell < k$, if E_k has a nontrivial zero (meaning a zero other than i and $e^{2\pi i/3}$), then the nontrivial zero of E_k closest to i is distinct from every zero of E_ℓ (see Lemma 2.4). We use a novel application of these methods to completely classify all monomial relations between Eisenstein series, as in the following theorem.

Theorem 1.1. *The equation*

$$\prod_{i=1}^{n} E_{k_i} = \prod_{j=1}^{m} E_{\ell_j}$$

with $k_i \neq \ell_j$ for any $1 \leq i \leq n$, $1 \leq j \leq m$ holds if and only if $k_i, \ell_j \in \{4, 6, 8, 10, 14\}$ for all i, j, and both sides can be rewritten as the same product of powers of E_4 and E_6 by using equation (1.1).

Thus there are no "nontrivial" monomial relations between Eisenstein series, where by nontrivial we mean relations which cannot be immediately obtained by taking identical products and augmenting them with the relations in equation (1.1).

Finally, as an application of Theorem 1.1, we will show in Theorem 3.4 a certain inequality between critical values of the L-functions associated to normalized Hecke eigenforms.

2. Finding a distinct zero of E_k

To prove Theorem 1.1, we must show that, with the exception of E_4, E_6, E_8, E_{10}, and E_{14}, every Eisenstein series has at least one zero not shared by any Eisenstein series of lesser weight. In Lemma 2.2, we find a bound on the zeros $\alpha^*_{k,n}$ of F_k in terms of their approximate values $\alpha_{k,n}$ (see Definition 2.1). Corollary 2.3 to this lemma gives an interesting result regarding the zeros of Eisenstein series, helpful in proving Lemma 2.4. Lemma 2.4 states that the nontrivial zero of E_k closest to i, if it exists, is distinct from every zero of E_ℓ for $\ell < k$, providing us with what we need to prove Theorem 1.1.

We begin by introducing a useful definition.

Definition 2.1 *([Noz08]).* Let $\alpha_{k,n}$ refer to the nth-zero of $2\cos(k\theta/2)$ located at $\pi\left(\frac{1}{2} + \frac{2n-1}{k}\right)$ if $k \equiv 0 \pmod 4$ and at $\pi\left(\frac{1}{2} + \frac{2n}{k}\right)$ if $k \equiv 2 \pmod 4$, for $1 \leq n \leq \dim(M_k) - 1$ where M_k is the space of modular forms of weight k and level one. Let $\alpha^*_{k,n}$ refer to the unique nth zero of $F_k(\theta)$ approximated by $\alpha_{k,n}$, where $F_k(\theta)$ is defined in equation (1.2).

Note that the nontrivial zeros of E_k are exactly the points $e^{i\alpha^*_{k,n}}$.

The following lemma is a slight improvement on the statement of Lemma 3.1 in [Noz08] which formalizes the sense in which $\alpha^*_{k,n}$ is "approximated" by $\alpha_{k,n}$ for $\alpha_{k,n}$ sufficiently close to $\frac{\pi}{2}$. We will rely heavily upon it moving forwards.

Lemma 2.2. *For any real $c \geq 1$, there exists a positive integer K_c such that if $k \geq K_c$ and $\alpha_{k,n} + \frac{\pi}{ck^2} \leq 11\pi/18$ then*

$$\alpha_{k,n} - \frac{\pi}{ck^2} < \alpha^*_{k,n} < \alpha_{k,n} + \frac{\pi}{ck^2}.$$

Proof. Notice that

$$\begin{aligned}\left|2\cos\left(\frac{k}{2}(\alpha_{k,n} \pm \frac{\pi}{ck^2})\right)\right| &= \left|2\cos\left(\frac{k}{2}\alpha_{k,n}\right)\cos\left(\pm\frac{\pi}{2ck}\right) - 2\sin\left(\frac{k}{2}\alpha_{k,n}\right)\sin\left(\pm\frac{\pi}{2ck}\right)\right| \\ &= 2\sin\left(\frac{\pi}{2ck}\right).\end{aligned}$$

So we can write

$$\begin{aligned}\left|2\cos\left(\frac{k}{2}\left(\alpha_{k,n} \pm \frac{\pi}{ck^2}\right)\right)\right| - \left|T_k\left(\alpha_{k,n} \pm \frac{\pi}{ck^2}\right)\right| &= 2\sin\left(\frac{\pi}{2ck}\right) - \left|T_k\left(\alpha_{k,n} \pm \frac{\pi}{ck^2}\right)\right| \\ &> 2\frac{2}{\pi}\frac{\pi}{2ck} - \frac{3}{2}\left(\frac{1}{1.1}\right)^k \\ &= \frac{4(1.1)^k - 3ck}{2ck(1.1)^k}.\end{aligned}$$

For clarification on the inequality, see equation (3.5) in [Noz08]. We now have

$$\begin{aligned}2\sin\left(\frac{\pi}{2ck}\right) - \left|T_k\left(\alpha_{k,n} \pm \frac{\pi}{ck^2}\right)\right| > 0 \ \text{ if } \ 4(1.1)^k - 3ck > 0, \\ \text{i.e. } \ 1.1^k k^{-1} > \frac{3c}{4}.\end{aligned}$$

The expression $1.1^k k^{-1}$ is unbounded and strictly increasing for $k \in [1, \infty)$. Let K_c be the minimum positive integer such that $1.1^{K_c} K_c^{-1} > \frac{3c}{4}$ and let $k \geq K_c$. Then

$$2\sin\left(\frac{\pi}{2ck}\right) > \left|T_k\left(\alpha_{k,n} \pm \frac{\pi}{ck^2}\right)\right|.$$

Therefore F_k is nonzero at $\alpha_{k,n} \pm \frac{\pi}{ck^2}$, and the sign of $F_k\left(\alpha_{k,n} \pm \frac{\pi}{ck^2}\right)$ is determined by the sign of $2\cos\left(\frac{k}{2}\left(\alpha_{k,n} \pm \frac{\pi}{ck^2}\right)\right)$. Since the points $\alpha_{k,n} \pm \frac{\pi}{ck^2}$ are symmetric around a zero of $2\cos(k\theta/2)$, we know that

$$2\cos\left(\frac{k}{2}\left(\alpha_{k,n} + \frac{\pi}{ck^2}\right)\right) = -2\cos\left(\frac{k}{2}\left(\alpha_{k,n} - \frac{\pi}{ck^2}\right)\right),$$

and thus

$$F_k\left(\alpha_{k,n} + \frac{\pi}{ck^2}\right) F_k\left(\alpha_{k,n} - \frac{\pi}{ck^2}\right) < 0.$$

This implies that F_k changes signs on this interval, so there must exist an $\alpha'_{k,n}$ satisfying

$$\alpha_{k,n} - \frac{\pi}{ck^2} < \alpha'_{k,n} < \alpha_{k,n} + \frac{\pi}{ck^2}$$

and $F_k(\alpha'_{k,n}) = 0$. Now we recall from Rankin and Swinnerton-Dyer [RS70] that $\alpha^*_{k,n}$ is the unique zero in the interval $(\alpha_{k,n} - \frac{\pi}{k}, \alpha_{k,n} + \frac{\pi}{k})$. Since $c \geq 1$, we have that $\frac{\pi}{ck^2} < \frac{\pi}{k}$. Thus we know that $\alpha'_{k,n} = \alpha^*_{k,n}$, completing the proof. □

Note that K_c increases only logarithmically with respect to c. For instance, $K_1 = 34, K_2 = 44, K_{10} = 65, K_{10^3} = 120, K_{10^6} = 198$, and so on. Thus, this is an efficient way to bound the error between $\alpha_{k,n}$ and $\alpha^*_{k,n}$ for surprisingly small k. Additionally, the above proof makes it clear that we could replace $\frac{\pi}{ck^2}$ with the bound $\frac{\pi}{ck^2}e^{-dk}$ for any d such that $e^d < 1.1$ due to the dependence on the exponential term 1.1^k. However, this minor improvement is unnecessary for the proof of the main theorem.

Remark. It is important for the proofs of Corollary 2.3 and Lemma 2.4 that Lemma 2.2 applies to the zeros $\alpha_{k,1}$ for all $k \geq 34$, and also applies to $\alpha_{k,2}$ for $k \geq 36$ and $k \equiv 0 \pmod 4$. Under these conditions we may choose $c \geq 1$ with corresponding $K_c \geq 34$ (noting that $K_1 = 34$), and for $j = 1$ or $j = 2$ we have

$$\alpha_{k,j} + \frac{\pi}{ck^2} \leq \pi\left(\frac{1}{2} + \frac{3}{k}\right) + \frac{\pi}{k^2} < \frac{11\pi}{18}.$$

Therefore, the condition $\alpha_{k,j} + \frac{\pi}{ck^2} \leq \frac{11\pi}{18}$ required for Lemma 2.2 is satisfied under these conditions.

Lemma 2.2 implies the following property of the zeros of F_k, which will aid in the proof of Lemma 2.4.

Corollary 2.3. *The sequences*

$$\{\alpha^*_{4j,1}\}_{j\geq 3}, \quad \{\alpha^*_{4j+2,1}\}_{j\geq 4}$$

are each strictly decreasing to $\frac{\pi}{2}$.

Proof. First, let $k = 4j$ and $k \geq 36$. We choose $c = 1$ and apply Lemma 2.2 to get

$$\alpha_{k,1} - \frac{\pi}{k^2} < \alpha^*_{k,1} < \alpha_{k,1} + \frac{\pi}{k^2}.$$

This yields the relation

$$\begin{aligned}\alpha^*_{k,1} - \alpha^*_{k+4,1} &> \alpha_{k,1} - \alpha_{k+4,1} - \pi\left(\frac{1}{k^2} + \frac{1}{(k+4)^2}\right) \\ &= \pi\frac{k+4-k}{k(k+4)} - \pi\frac{(k+4)^2 + k^2}{k^2(k+4)^2} \\ &= \pi\frac{2k^2 + 8k - 16}{k^2(k+4)^2}.\end{aligned}$$

Therefore,

$$\alpha^*_{k,1} > \alpha^*_{k+4,1} \text{ if } 2k^2 + 8k - 16 > 0,$$

which is true for $k \geq 36$. Likewise, for $k = 4j + 2$ and $k \geq 34$,

$$\begin{aligned}\alpha^*_{k,1} - \alpha^*_{k+4,1} &> \pi\frac{2k + 8 - 2k}{k(k+4)} - \pi\frac{(k+4)^2 + k^2}{k^2(k+4)^2} \\ &= \pi\frac{6k^2 + 24k - 16}{k^2(k+4)^2}.\end{aligned}$$

Therefore,

$$\alpha^*_{k,1} > \alpha^*_{k+4,1} \text{ if } 6k^2 + 24k - 16 > 0,$$

which is true for $k \geq 34$. When $k < 34$, the relation $\alpha^*_{k,1} > \alpha^*_{k+4,1}$ can be computationally verified (see Table 4 and Table 5). □

Remark. This corollary can be generalized to the statement that for any fixed m and for j sufficiently large, the sequences $\{\alpha^*_{4j,m}\}$ and $\{\alpha^*_{4j+2,m}\}$ are each strictly decreasing to $\frac{\pi}{2}$. However, the exact formulation of this proof is tedious as it requires that $k = 4j$ or $4j + 2$ be large enough relative to m to ensure that $\alpha_{k,m} + \frac{\pi}{ck^2} < \frac{11\pi}{18}$. In general, this will not happen for the first few k for which E_k has an mth zero on A.

The following Lemma 2.4 is the key component in the proof of Theorem 1.1. The proof of Lemma 2.4 proceeds in a very similar fashion to that of Corollary 2.3.

Lemma 2.4. *If E_k has a nontrivial zero, then the nontrivial zero of E_k closest to i is distinct from every zero of E_ℓ for all $\ell < k$.*

Proof. We carry out the proof in 6 cases according to the values of k and ℓ.

Case 1. Let $k \equiv \ell \pmod 4$ and suppose that $\ell < k$. Then Corollary 2.3 tells us that

$$\alpha^*_{k,1} < \alpha^*_{\ell,1} < \alpha^*_{\ell,m}$$

for $m > 1$, completing the proof for this case.

Case 2. Let $k, \ell \geq 34$, $k \equiv 0 \pmod 4$, $\ell \equiv 2 \pmod 4$, and $\ell < k$. We choose $c = 1$ and apply Lemma 2.2 to get

$$\alpha^*_{\ell,1} - \alpha^*_{k,1} > \alpha_{\ell,1} - \alpha_{k,1} - \pi\left(\frac{1}{\ell^2} + \frac{1}{k^2}\right)$$

$$
\begin{aligned}
&= \pi \frac{2k-\ell}{k\ell} - \pi \frac{k^2+\ell^2}{(k\ell)^2} \\
&= \pi \frac{2k^2\ell - \ell^2 k - k^2 - \ell^2}{(k\ell)^2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\alpha^*_{\ell,1} - \alpha^*_{k,1} > 0 \ \text{ if } \ 2k^2\ell - \ell^2 k - k^2 - \ell^2 &> 2k^2\ell - (\ell+2)k^2 > 0, \\
\iff \ell - 2 &> 0.
\end{aligned}
$$

So we have

$$
\alpha^*_{k,1} < \alpha^*_{\ell,1} < \alpha^*_{\ell,m}
$$

for $m > 1$, completing the proof for this case.

Case 3. Let $k, \ell \geq 38$, $k \equiv 2 \pmod 4$, $\ell \equiv 0 \pmod 4$, and $\ell < k/2$. We choose $c = 1.3$, corresponding to $K_{1.3} = 38$, and apply Lemma 2.2 to get

$$
\begin{aligned}
\alpha^*_{\ell,1} - \alpha^*_{k,1} &> \alpha_{\ell,1} - \alpha_{k,1} - \pi \left(\frac{1}{1.3\ell^2} + \frac{1}{1.3k^2} \right) \\
&= \pi \frac{k-2\ell}{k\ell} - \pi \frac{k^2+\ell^2}{1.3(k\ell)^2} \\
&= \pi \frac{1.3k^2\ell - 2.6\ell^2 k - k^2 - \ell^2}{1.3(k\ell)^2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\alpha^*_{\ell,1} - \alpha^*_{k,1} > 0 \ \text{ if } \ 1.3k^2\ell - 2.6\ell^2 k - k^2 - \ell^2 > 0, \\
\text{i.e.} \quad -\left(\frac{k}{\ell}\right)^2 + 1.3k\left(\frac{k}{\ell}\right) - (2.6k+1) > 0.
\end{aligned}
$$

The parabola $-x^2 + 1.3kx - (2.6k+1)$ faces downwards. It can be checked that this parabola is positive at $x = \frac{k}{(k-2)/2}$ and at $x = \frac{k}{2}$ when $k \geq 38$, and thus it is positive on $\left[\frac{k}{(k-2)/2}, \frac{k}{2}\right]$. This tells us that

$$
\alpha^*_{k,1} < \alpha^*_{\ell,1} < \alpha^*_{\ell,m}
$$

for $m > 1$, completing the proof for this case.

Case 4. Let $k, \ell \geq 38$, $k \equiv 2 \pmod 4$, $\ell \equiv 0 \pmod 4$, and $\ell > k/2$. We again choose $c = 1.3$ and apply Lemma 2.2 to get

$$\begin{aligned}\alpha^*_{\ell,1} - \alpha^*_{k,1} &< \alpha_{\ell,1} - \alpha_{k,1} + \pi\left(\frac{1}{1.3\ell^2} + \frac{1}{1.3k^2}\right)\\ &= \pi\frac{k-2\ell}{k\ell} + \pi\frac{\ell^2+k^2}{1.3(k\ell)^2}\\ &= \pi\frac{1.3k^2\ell - 2.6\ell^2 k + k^2 + \ell^2}{1.3(k\ell)^2}.\end{aligned}$$

Thus,

$$\begin{aligned}\alpha^*_{\ell,1} - \alpha^*_{k,1} < 0 \ \text{ if } \ &1.3k^2\ell - 2.6\ell^2 k + k^2 + \ell^2 < 0,\\ \text{i.e. } \ &\left(\frac{k}{\ell}\right)^2 + 1.3k\left(\frac{k}{\ell}\right) - (2.6k-1) < 0.\end{aligned}$$

The parabola $x^2+1.3kx-(2.6k-1)$ faces upwards. It can be checked that this parabola is negative at $x = \frac{k}{(k+2)/2}$ and at $x = \frac{k}{k-2}$ for $k \geq 38$. Thus it is negative on $\left[\frac{k}{k-2}, \frac{k}{(k+2)/2}\right]$, implying that

$$\alpha^*_{\ell,1} < \alpha^*_{k,1}.$$

Next, we revert to $c = 1$ from Lemma 2.2 and compute

$$\begin{aligned}\alpha^*_{\ell,2} - \alpha^*_{k,1} &> \alpha_{\ell,2} - \alpha_{k,1} - \pi\left(\frac{1}{\ell^2} + \frac{1}{k^2}\right)\\ &= \pi\frac{3k-2\ell}{k\ell} - \pi\frac{\ell^2+k^2}{(k\ell)^2}\\ &= \pi\frac{3k^2\ell - 2\ell^2 k - k^2 - \ell^2}{(k\ell)^2}.\end{aligned}$$

Thus,

$$\begin{aligned}\alpha^*_{\ell,2} - \alpha^*_{k,1} > 0 \ \text{ if } \ &3k^2\ell - 2\ell^2 k - k^2 - \ell^2 > 3k^2\ell - (2\ell+2)k^2 > 0,\\ \iff &\ell - 2 > 0.\end{aligned}$$

So we have

$$\alpha^*_{\ell,1} < \alpha^*_{k,1} < \alpha^*_{\ell,2} < \alpha^*_{\ell,m}$$

for $m > 2$, completing the proof for this case.

Case 5. Let $k > 72, \ell \leq 36$. We choose $c = 1$ and apply Lemma 2.2 to get

$$\alpha^*_{k,1} < \pi\left(\frac{1}{2} + \frac{2}{k}\right) + \frac{\pi}{k^2} < 1.657.$$

Then since $1.657 < \alpha^*_{\ell,1} < \alpha^*_{\ell,m}$ for $m > 1$ when $\ell \leq 36$, the lemma is proven for $k > 72, \ell \leq 36$.

Case 6. In the final remaining case, when $k \leq 72, \ell \leq 36$, the relations $\alpha_{k,1} \neq \alpha_{\ell,1}$, $\alpha_{k,1} \neq \alpha_{\ell,2}$ can be verified with the data in Tables 1, 2, and 3. Note that we need not worry about $\alpha^*_{36,3}$ because the closest zero of F_{36} to $\alpha^*_{k,1}$ must be $\alpha^*_{36,1}$ or $\alpha^*_{36,2}$. □

Remark. Using the same method shown above, the result of this lemma could easily be extended to the case where $\ell > k$, assuming ℓ is sufficiently close to k. In particular, this result is still true if $\ell > k$ and $\dim(M_k) \geq \dim(M_\ell)$, as when ℓ and k are sufficiently close, we have an interlacing property whereby either $\alpha^*_{k,1} < \alpha^*_{\ell,1} < \alpha^*_{k,2}$ or $\alpha^*_{\ell,1} < \alpha^*_{k,1} < \alpha^*_{\ell,2}$.

3. Conclusion and discussion

We are now ready to present the following proof of Theorem 1.1.

Proof of Theorem 1.1. It is clear that if both sides can be written as the same product of powers of E_4 and E_6 then the equation holds. Conversely, suppose

$$\prod_{i=1}^{n} E_{k_i} = \prod_{j=1}^{m} E_{\ell_j}$$

for some Eisenstein series $\{E_{k_i}\}_{i=1}^n$, $\{E_{\ell_j}\}_{j=1}^m$ satisfying $k_i \neq \ell_j$ for all i, j. Without loss of generality, assume that $k_n \geq k_i$ and $k_n > \ell_j$ for all i, j. Assume also that E_{k_n} has at least one non-trivial zero on the arc A, and let z_0 be the nontrivial zero of E_{k_n} closest to i. Then, from Lemma 2.4,

$$\prod_{i=1}^{n} E_{k_i}(z_0) = 0 \neq \prod_{j=1}^{m} E_{\ell_j}(z_0),$$

a contradiction, implying that $k_i, \ell_j \leq 14$ for all i, j. Assume $k_i = 12$ for some i. Since $\ell_j = 14$ or $\ell_j \leq 10$ for all j, this implies the same contradiction as above, and similarly if $\ell_j = 12$ for some j. Thus, every element of $\{E_{k_i}\}_{i=1}^n \cup \{E_{\ell_j}\}_{j=1}^m$ must also be an element of $\{E_4, E_6, E_8, E_{10}, E_{14}\}$, and can be written as products of powers of E_4 and E_6. Therefore we have

$$\prod_{i=1}^{n} E_{k_i} = \prod_{j=1}^{m} E_{\ell_j} = E_4^a E_6^b,$$

completing the proof of Theorem 1.1. □

Although computational difficulties have restricted our result to only the first zero of E_k, data for small k, ℓ as well as the methods in [Noz08] suggest the following conjecture.

Conjecture 3.1. *When $k \neq \ell$, every zero of E_k on A is distinct from every zero of E_ℓ on A.*

The methods of Lemma 2.4 could prove partial results towards this conjecture. However, these techniques only work for zeros in the interval $(\pi/2, 11\pi/18]$ due to $T_k(\theta)$ failing to have the same exponential decay bound on $(11\pi/18, 2\pi/3)$. Even on $(\pi/2, 11\pi/18]$, note that when ℓ divides k, the zeros of $2\cos(\ell\theta/2)$ are all included in the zeros of $2\cos(k\theta/2)$. This means that any argument made with the bounding strategy presented above will not suffice, and a closer look is needed at the exact behavior of $T_k(\theta)$.

This conjecture is closely related to the zero polynomials associated to each E_k.

Definition 3.2 *(Zero polynomial of E_k, [Gek01]).* Let

$$\varphi_k(x) = \prod_{i=1}^{n}(x - j(z_i))$$

where z_i runs over zeros of E_k other than i or $e^{\frac{2\pi i}{3}}$, and $j(z)$ is the j-invariant function.

The truth of the following conjecture would then be sufficient to imply Theorem 1.1.

Conjecture 3.3 *(Cornelissen [Cor99] and Gekeler [Gek01]). The zero polynomials $\varphi_k(x)$ are irreducible over $\mathbb{Q}$.*

It is not hard to show that distinct Eisenstein series have distinct zero polynomials, excluding E_4, E_6, E_8, E_{10}, and E_{14}, which all share the same trivial zero polynomial. Since no two distinct irreducible polynomials share common zeros, Conjecture 3.3 implies Conjecture 3.1, which then implies Theorem 1.1 analogously to the proof above.

Finally, let us discuss an application of Theorem 1.1 on the critical values of L-functions. Let $f \in S_k$ be a cuspform of weight k and level one, and let ℓ be even with $4 \leq \ell \leq k - 4$. The Rankin-Selberg convolution yields the following identity [Gha00, Section 2]

$$\langle f, E_\ell E_{k-\ell}\rangle = -\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \cdot \frac{2\ell}{B_\ell} \cdot \frac{L(k-1, f)L(k-\ell, f)}{\zeta(k-\ell)},$$

where $L(s, f)$ denotes the usual L-function associated to f.

If we write

$$E_\ell E_{k-\ell} = E_k + \sum_f c_{\ell,f} f$$

as a linear combination over a basis of normalized Hecke eigenforms, then

$$c_{\ell,f} = \frac{\langle f, E_\ell E_{k-\ell}\rangle}{\langle f, f\rangle}.$$

Note that if $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ is an automorphism of $\mathbb{C}$, then

$$c_{\ell,f^\sigma} = c_{\ell,f}^\sigma.$$

Now, suppose we are given two even numbers ℓ_1 and ℓ_2 with $4 \leq \ell_1 < \ell_2 \leq k-4$ and $\ell_1 + \ell_2 \neq k$. By Theorem 1.1, the inequality

$$E_{\ell_1} E_{k-\ell_1} \neq E_{\ell_2} E_{k-\ell_2}$$

holds except for

$$(\ell_1, \ell_2, k) = (4, 8, 18), (4, 10, 18), (6, 10, 20).$$

Let us assume the inequality holds. Then there exists a normalized Hecke eigenform f of weight k such that $c_{\ell_1,f} \neq c_{\ell_2,f}$, or equivalently

$$\langle f, E_{\ell_1} E_{k-\ell_1} \rangle \neq \langle f, E_{\ell_2} E_{k-\ell_2} \rangle.$$

Therefore,

$$\frac{2\ell_1}{B_{\ell_1}} \cdot \frac{L(k-\ell_1, f)}{\zeta(k-\ell_1)} \neq \frac{2\ell_2}{B_{\ell_2}} \cdot \frac{L(k-\ell_2, f)}{\zeta(k-\ell_2)}.$$

As

$$\zeta(k) = -\frac{(2\pi i)^k}{2(k!)} B_k$$

the inequality can be further simplified as

$$\frac{\ell_1 \cdot (k-\ell_1)!}{B_{\ell_1} B_{k-\ell_1}} \cdot \frac{L(k-\ell_1, f)}{(2\pi i)^{k-\ell_1}} \neq \frac{\ell_2 \cdot (k-\ell_2)!}{B_{\ell_2} B_{k-\ell_2}} \cdot \frac{L(k-\ell_2, f)}{(2\pi i)^{k-\ell_2}}.$$

If we assume Maeda's conjecture (Conjecture 1.2 in [HM97]) on the simplicity of the Hecke algebra on S_k, then the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts transitively on the basis of normalized eigenforms. Thus, if $c_{\ell_1,f} \neq c_{\ell_2,f}$ for some f, then

$$c_{\ell_1,f} \neq c_{\ell_2,f}$$

for all eigenforms f.

The above discussion can be summarized as the following result.

Theorem 3.4. *Suppose* $\dim(S_k) \geq 1$. *Let* ℓ_1 *and* ℓ_2 *be even numbers with* $4 \leq \ell_1 < \ell_2 \leq k-4$ *and* $\ell_1 + \ell_2 \neq k$ *such that*

$$(\ell_1, \ell_2, k) \neq (4, 8, 18), (4, 10, 18), (6, 10, 20).$$

Then there exists a normalized eigenform f of weight k such that

$$\frac{\ell_1 \cdot (k-\ell_1)!}{B_{\ell_1} B_{k-\ell_1}} \cdot \frac{L(k-\ell_1, f)}{(2\pi i)^{k-\ell_1}} \neq \frac{\ell_2 \cdot (k-\ell_2)!}{B_{\ell_2} B_{k-\ell_2}} \cdot \frac{L(k-\ell_2, f)}{(2\pi i)^{k-\ell_2}}.$$

Furthermore, if the Hecke algebra on S_k is simple, then the above inequality holds for every normalized eigenform of weight k.

See Table 6 for computational evidence of the above theorem for normalized eigenforms of weights $12, 16, \cdots, 24$ and 26.

Acknowledgments

This research was conducted during the 2020 REU program at Clemson University, supported by NSF-DMS Grant 1547399.

Appendix A. Tables

Tables 1, 2, and 3 complete the proof of Lemma 2.4 and were computed using the closed formula given in [Koh04]. The $\alpha^*_{k,n}$ and $\alpha_{k,n}$ values follow the same notation of 2.1. Tables 4 and 5 complete the proof of Corollary 2.3. For Table 6 let

$$\gamma_{k,\ell} = \frac{\ell \cdot (k-\ell)!}{B_\ell B_{k-\ell}} \cdot \frac{L(k-\ell, f)}{(2\pi i)^{k-\ell}},$$

so that the values can be used to verify Theorem 3.4 [LMFDB]. Calculations were computed with 1000 significant digits in Pari/GP [Bat+98]. See [Gri20] for source code.

Table 1
First Zero of F_k with $k \equiv 0 \pmod 4$.

k	$\alpha^*_{k,1}$	$\alpha_{k,1}$	$\lvert\alpha^*_{k,1} - \alpha_{k,1}\rvert$
12	1.824855600	1.832595715	0.007740115042
16	1.768610843	1.767145868	0.001464974861
20	1.727597772	1.727875959	0.0002781873219
24	1.701752889	1.701696021	$5.686819194 \times 10^{-5}$
28	1.682984081	1.682996064	$1.198326671 \times 10^{-5}$
32	1.668973688	1.668971097	$2.591185736 \times 10^{-6}$
36	1.658062219	1.658062789	$5.705771539 \times 10^{-7}$
40	1.649336271	1.649336143	$1.274571294 \times 10^{-7}$
44	1.642196131	1.642196160	$2.879756905 \times 10^{-8}$
48	1.636246180	1.636246174	$6.567376166 \times 10^{-9}$
52	1.631211569	1.631211570	$1.508052409 \times 10^{-9}$
56	1.626896196	1.626896196	$3.753475860 \times 10^{-10}$
60	1.623156205	1.623156204	$4.270295682 \times 10^{-10}$
64	1.619883716	1.619883712	$3.720879568 \times 10^{-9}$
68	1.616996237	1.616996219	$1.869869402 \times 10^{-8}$
72	1.614429460	1.614429558	$9.808943303 \times 10^{-8}$

Table 2
First Zero of F_k with $k \equiv 2 \pmod 4$.

k	$\alpha^*_{k,1}$	$\alpha_{k,1}$	$\lvert\alpha^*_{k,1} - \alpha_{k,1}\rvert$
18	1.915434107	1.919862177	0.004428069756
22	1.857250367	1.856395659	0.0008547076081
26	1.812293607	1.812457300	0.0001636936144
30	1.780269358	1.780235837	$3.352128264 \times 10^{-5}$
34	1.755588829	1.755595895	$7.065884793 \times 10^{-6}$
38	1.736144836	1.736143309	$1.526948311 \times 10^{-6}$
42	1.720395641	1.720395977	$3.359360348 \times 10^{-7}$
46	1.707387387	1.707387312	$7.496661428 \times 10^{-8}$
50	1.696460016	1.696460033	$1.692060382 \times 10^{-8}$
54	1.687151614	1.687151610	$3.854938813 \times 10^{-9}$
58	1.679127107	1.679127108	$8.851164162 \times 10^{-10}$
62	1.672138026	1.672138025	$2.045891951 \times 10^{-10}$
66	1.665996104	1.665996104	$4.776831270 \times 10^{-11}$
70	1.660556117	1.660556117	$1.832176137 \times 10^{-11}$

Table 3
Second Zero of F_k.

k	$\alpha^*_{k,2}$	$\alpha_{k,2}$	$\lvert\alpha^*_{k,2} - \alpha_{k,1}\rvert$
24	1.960354810	1.963495408	0.003140598274
28	1.907999656	1.907395540	0.0006041163696
30	1.987251378	1.989675347	0.002423969265
32	1.865205828	1.865320638	0.0001148096729
34	1.940858142	1.940395463	0.0004626798557
36	1.832618984	1.832595715	$2.326963148 \times 10^{-5}$

Table 4
$\alpha^*_{k,1}$ Range for $k \equiv 0 \pmod 4$.

k	$(\alpha_{k,1} - \frac{\pi}{k^2})$	$(\alpha_{k,1} + \frac{\pi}{k^2})$
12	1.810779099	1.854412330
16	1.754874021	1.779417714
20	1.720021978	1.735729941
24	1.696241867	1.707150175
28	1.678988931	1.687003198
32	1.665903136	1.672039059

Table 5
$\alpha^*_{k,1}$ Range for $k \equiv 2 \pmod 4$.

k	$(\alpha_{k,1} - \frac{\pi}{k^2})$	$(\alpha_{k,1} + \frac{\pi}{k^2})$
14	2.003566743	2.035623811
18	1.910165904	1.929558451
22	1.849904765	1.862886553
26	1.807809974	1.817104627
30	1.776745179	1.783726496
34	1.752878254	1.758313535

Table 6
$\gamma_{k,l}$-Values for Small k.

k	ℓ	$\gamma_{k,\ell}$
12	4	55.61565697
	6	−98.10601890
16	4	62.82185616
	6	−146.9258242
	8	181.7264369
	10	−146.9258242
18	4	58.03385626
	6	−109.4290547
	8	58.03385626
	10	58.03385626
	12	−109.4290547
20	4	60.48221866
	6	−130.6645810
	8	142.2118725
	10	−130.6645810
	12	142.2118725
	14	−130.6645810
22	4	59.91248008
	6	−125.0255289
	8	114.1435704
	10	−42.34605190
	12	−42.34605190
	14	114.1435704
	16	−125.0255289
24.a	4	59.77727589
	6	−124.2114447
	8	113.9746791
	10	−57.30971502
	12	28.89997856
	14	−57.30971502
	16	113.9746791
	18	−124.2114447
24.b	4	60.29164885
	6	−128.4638733
	8	129.6435717
	10	−89.53486445
	12	69.51387145
	14	−89.53486445
	16	129.6435717
	18	−128.4638733
26	4	59.99879213
	6	−125.9822665
	8	119.8374044
	10	−64.87258915
	12	17.70494990
	14	17.70494990
	16	−64.87258915
	18	119.8374044
	20	−125.9822665

References

[Bat+98] C. Batut, et al., User's guide to PARI-GP, see also http://pari.home.ml.org, by anonymous ftp from ftp://megrez.math.u-bordeaux.fr/pub/pari, 1998.

[Cor99] Gunther Cornelissen, Zeros of Eisenstein series, quadratic class numbers and supersingularity for rational function fields, Math. Ann. 314 (1999) 175–196.

[Duk99] W. Duke, When is the product of two Hecke eigenforms and eigenform?, in: Number Theory in Progress, vol. 2, 1999, pp. 737–741.

[EL07] B. Emmons, D. Lanphier, Products of an arbitrary number of Hecke eigenforms, Acta Arith. 130 (2007) 311–319.

[Gek01] Ernst-Ulrich Gekeler, Some observations on the arithmetic of Eisenstein series of the modular group $SL(2, \mathbb{Z})$, Arch. Math. 77 (2001) 5–21.

[Gha00] E. Ghate, On monomial relations between Eisenstein series, J. Ramanujan Math. Soc. 15 (2000) 71–79.

[Gri20] T. Griffin, Zeros of Eisenstein series, Version 2.0, https://doi.org/10.5281/zenodo.3887249, June 2020.

[HM97] Haruzo Hida, Yoshitaka Maeda, Non-Abelian base change for totally real fields, Pac. J. Math. 181 (1997) 189–217.

[Koh04] W. Kohnen, Zeroes of Eisenstein series, Kyushu J. Math. 58 (2004) 251–256.

[LMFDB] The LMFDB Collaboration, The L-functions and modular forms database, http://www.lmfdb.org, 2020. (Accessed 9 June 2020).

[Noz08] H. Nozaki, A separation property of the zeros of Eisenstein series for $\mathrm{SL}(2, \mathbb{Z})$, Bull. Lond. Math. Soc. 40 (2008) 26–36.

[RS70] F.K.C. Rankin, H.P.F. Swinnerton-Dyer, On the zeros of Eisenstein series, Bull. Lond. Math. Soc. 2 (1970) 169–170.